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Shifted large-N expansion for the power-law potentials in the Klein–Gordon equation with applications

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Abstract. Analytic expressions for the spectrum and the wavefunction resulting from the relativistic generalisation of the shifted large-N expansion method as applied to the general power-law potentials $V(r) = B + Ar^{\nu}$ are presented. These expressions obtained in the context of the Klein-Gordon equation are then used to compute the mass spectra and leptonic decay widths of mesonic states composed of heavy quarks. The relativistic corrections are found to be in good agreement with those obtained by elaborate analytic (WKB) and numerical methods. Compact analytic results of the shifted large-N expansion for relativistic systems are seen to be applicable to a much wider class of problems than are most other approximation methods.

1. Introduction

The so-called large-N expansion method, where N is the number of spatial dimensions, provides a powerful, systematic and analytic technique for determining energy eigenvalues and eigenfunctions of a variety of non-relativistic potential problems [1, 2]. Although the method involves expansion in terms of a parameter $1/K$ where $K = N + 2l$, it is non-perturbative in the sense that it does not involve expansion in terms of coupling constants contained in the potential. The convergence of this method is rather slow, particularly for the s states. This difficulty has been subsequently circumvented by Sukhatme and Imbo [3] who suggested a change in the expansion parameter from $1/K$ to $1/\bar{K}$ where $\bar{K} = N + 2l - a$ and the shift parameter a was chosen in such a manner that the exact analytic expressions for the bound-state energies for the Coulomb and harmonic oscillator potentials are restored. The accuracy of the shifted $1/N$ expansion technique has been tested for a variety of spherically symmetric potentials [4–8] which have applications in different areas of physics.

It is natural to expect that the large-N expansion method should also be useful for relativistic bound-state problems. An extension of the unshifted large-N expansion method to spherically symmetric relativistic potentials has recently been made by Miramontes and Pajares [9] and Chatterjee [10]. In connection with the Klein-Gordon (KG) equation these authors found that the relativistic corrections to the non-relativistic limit is non-leading in the $1/N$ expansion. However, the rate of convergence of the expansion is very slow for the relativistic part of the energy eigenvalue as compared to that for the non-relativistic part.

Recently we have shown [11] that the shifted $1/N$ method can also be extended to the relativistic wave equation for the Coulomb problem. The attractive feature of our work is that the convergence of the relativistic part of the $1/N$ expansion is faster than the corresponding one obtained in the unshifted framework [9, 10]. Stimulated by this observation, we propose here to examine the shifted $1/N$ expansion technique for the KG equation for the general power-law potentials with a view to studying the relativistic effects in hadron spectroscopy. The main trick of our approach lies in the fact that it is possible to convert the KG equation to a Schrödinger-like equation to which one may subsequently apply the method of [4], with only one difference that one must fix up the 'shift parameter' in a different way. Analogous to the non-relativistic calculation [4], we obtain closed analytic expressions for the relativistic part (of the order $1/c^2$) of the binding energy and leading-order wavefunction.

The plan of this paper is as follows. In § 2, the method of reducing the KG equation for a spherically symmetric potential to a Schrödinger-like equation is presented and the effective potential is established. We also give explicit expressions for the total binding energy including relativistic corrections correct up to order $1/c^2$ and the leading-order wavefunction. For specific applications, we present in § 3 the analytic results for the mass spectra of $q\bar{q}$ systems for the power-law potentials of the form $V(r)=B+Ar^{\nu}$. The leading-order wavefunction is also presented for computation of relativistic effects to leptonic decay widths of various quarkonia. We compare our results (using the shifted $1/N$ method) with those of other authors for a variety of potentials in three spatial dimensions. A good agreement is observed in general. However, for the decay width calculation, we find that harmonic and linear confining potentials give worse results than those obtained in the Martin potential. In the concluding section, we make a few remarks about the advantage and usefulness of the present approach to other areas of physics.

2. Shifted $1/N$ expansion for a spin-zero relativistic particle

In this section we will formulate the shifted $1/N$ expansion of binding energy and the leading-order wavefunction for a particle moving in a spherically symmetric potential

$$V(r)=Ar^{\nu}+B \tag{1}$$

in the KG equation

$$(E-V)^2\psi=[-c^2\hbar^2\nabla^2+m^2c^4]\psi. \tag{2}$$

As a first step, one must reduce the radial part of the KG equation to that of the Schrödinger-like equation. The radial part of (2) in N-dimensional hyper-spherical coordinates for a particle of rest mass m in the potential $V(r)$ is

$$\left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{\hbar^2}{8mr^2}(k-1)(k-3)-\frac{1}{2mc^2}\{[E-V(r)]^2-m^2c^4\}\right)u(r)=0 \tag{3}$$

where $k=N+2l$.

Introducing a shift parameter a through the relation

$$\bar{k}=k-a \tag{4}$$

equation (3) becomes

$$\left\{-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\bar{k}^2\left[\frac{\hbar^2}{8mr^2}\left(1-\frac{1-a}{\bar{k}}\right)\left(1-\frac{3-a}{\bar{k}}\right)-\frac{1}{2mc^2Q}[(E-V(r))^2-m^2c^4]\right]\right\}u(r)=0 \tag{5}$$

where Q is a constant which rescales the potential. Both Q and a will be determined later. Following [9] the leading-order energy E_0 is given by

$$E_0 = V(r_0) + mc^2(1 + \hbar^2 Q/4m^2c^2r_0^2)^{1/2} \tag{6}$$

where r_0, the location of the minimum of the effective potential, is determined from

$$r_0^3 \frac{\mathrm{d}V}{\mathrm{d}r}\bigg|_{r=r_0} \left(1 + \frac{\hbar^2 Q}{4m^2c^2r_0^2}\right)^{1/2} = \frac{\hbar^2 Q}{4m}. \tag{7}$$

In [11] we have discussed the method for reducing (5) to a Schrödinger-like equation. We thus obtain

$$\left\{-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \bar{k}^2\left[\frac{\hbar^2}{8mr^2}\left(1 - \frac{1-a}{\bar{k}}\right)\left(1 - \frac{3-a}{\bar{k}}\right) + \frac{U(r)}{Q}\right]\right\} u(r) = \mathscr{E}u(r) \tag{8}$$

in which

$$U(r) = -\frac{1}{2mc^2}[(E_0 - V(r))^2 - m^2c^4] \tag{9}$$

$$\mathscr{E} = \frac{1}{2mc^2}[(E - V(r_0))^2 - (E_0 - V(r_0))^2]. \tag{10}$$

Clearly $U(r)$ and $\mathscr{E}$ play the role of the potential and the bound-state energy respectively for an effective non-relativistic problem and so one may apply the previous $1/\bar{k}$ expansion scheme [4] directly. However, the calculation is bound to be complicated and lengthy due to the complex structure of $U(r)$ and $\mathscr{E}$ as given in (9) and (10). For brevity, we give here only the final results:

$$\mathscr{E} = \frac{a^{(0)}\bar{k}^2}{r_0^2} + \frac{\Lambda\bar{k}}{r_0^2} + \frac{1}{r_0^2}(b^{(0)} + b^{(1)} + b^{(2)}) + \frac{1}{r_0^2\bar{k}}(c^{(1)} + c^{(2)} + c^{(3)} + c^{(4)}) + \ldots \tag{11}$$

and the leading-order bound-state wavefunction

$$U_{n,l}^{(0)}(r) \propto r^{(k-1)/2} \exp\left[-\lambda\left(\frac{r}{r_0}\right)^{\tilde{\omega}}\right] L_{n_r}^{(k-2)/\tilde{\omega}}\left[2\lambda\left(\frac{r}{r_0}\right)^{\tilde{\omega}}\right] \tag{12}$$

in which

$$\tilde{\omega} = \frac{2m}{\hbar^2}\omega = \frac{2m}{\hbar^2}\left(\frac{3\hbar^2}{4m^2} + \frac{r_0^4 U''(r_0)}{mQ}\right)^{1/2} \tag{13}$$

$$\frac{\bar{k}\Lambda}{r_0^2} = \frac{\bar{k}}{r_0^2}\left(-\frac{(2-a)\hbar^2}{4m} + (1+2n_r)\frac{\hbar\omega}{2}\right) \tag{14}$$

$$a^{(0)} = \frac{\hbar^2}{8m} + \frac{r_0^2 U(r_0)}{Q} \tag{15a}$$

$$b^{(0)} = \frac{(1-a)(3-a)\hbar^2}{8m} \tag{15b}$$

$$b^{(1)} = (1+2n_r)\tilde{\varepsilon}_2 + 3(1+2n_r+2n_r^2)\tilde{\varepsilon}_4 \tag{15c}$$

$$b^{(2)} = -\frac{1}{\hbar\omega}[\tilde{\varepsilon}_1^2 + 6(1+2n_r)\tilde{\varepsilon}_1\tilde{\varepsilon}_3 + (11+30n_r+30n_r^2)\tilde{\varepsilon}_3^2] \tag{15d}$$

$$c^{(1)} = (1+2n_r)\tilde{\delta}_2 + 3(1+2n_r+2n_r^2)\tilde{\delta}_4 + 5(3+8n_r+6n_r^2+4n_r^3)\tilde{\delta}_6 \tag{15e}$$

$$\begin{aligned} c^{(2)} = -\frac{1}{\hbar\omega}[&(1+2n_r)\tilde{\varepsilon}_2^2 + 12(1+2n_r+2n_r^2)\tilde{\varepsilon}_2\tilde{\varepsilon}_4 + 6(1+2n_r)\tilde{\varepsilon}_1\tilde{\delta}_3 \\ &+ 2(21+59n_r+51n_r^2+34n_r^3)\tilde{\varepsilon}_4^2 + 2\tilde{\varepsilon}_1\tilde{\delta}_1 + 30 \\ &\times(1+2n_r+2n_r^2)\tilde{\varepsilon}_1\tilde{\delta}_5 + 6(1+2n_r)\tilde{\varepsilon}_3\tilde{\delta}_1 + 2(11+30n_r+30n_r^2) \\ &\times\tilde{\varepsilon}_3\tilde{\delta}_3 + 10(13+40n_r+42n_r^2+28n_r^3)\tilde{\varepsilon}_3\tilde{\delta}_5] \end{aligned} \tag{15f}$$

$$\begin{aligned} c^{(3)} = \frac{1}{(\hbar\omega)^2}[&4\tilde{\varepsilon}_1^2\tilde{\varepsilon}_2 + 36(1+2n_r)\tilde{\varepsilon}_1\tilde{\varepsilon}_2\tilde{\varepsilon}_3 + 8(11+30n_r+30n_r^2)\tilde{\varepsilon}_2\tilde{\varepsilon}_3^2 \\ &+ 12(57+189n_r+225n_r^2+150n_r^3)\tilde{\varepsilon}_3^2\tilde{\varepsilon}_4 + 24(1+2n_r)\tilde{\varepsilon}_1^2\tilde{\varepsilon}_4 \\ &+ 8(31+78n_r+78n_r^2)\tilde{\varepsilon}_1\tilde{\varepsilon}_3\tilde{\varepsilon}_4] \end{aligned} \tag{15g}$$

$$\begin{aligned} c^{(4)} = -\frac{1}{(\hbar\omega)^3}[&8\tilde{\varepsilon}_1^3\tilde{\varepsilon}_3 + 108(1+2n_r)\tilde{\varepsilon}_1^2\tilde{\varepsilon}_3^2 + 48(11+30n_r+30n_r^2)\tilde{\varepsilon}_1\tilde{\varepsilon}_3^3 \\ &+ 30(31+109n_r+141n_r^2+94n_r^3)\tilde{\varepsilon}_3^4] \end{aligned} \tag{15h}$$

and

$$\tilde{\varepsilon}_j = \frac{\varepsilon_j}{(2m\omega/\hbar)^{j/2}} \qquad \tilde{\delta}_j = \frac{\delta_j}{(2m\omega/\hbar)^{j/2}}. \tag{16}$$

The expressions for ε_j and δ_j are the same as in [4] with the exception that $V(r)$ and its derivatives are to be replaced by $U(r)$ and its corresponding derivatives respectively. Here n_r stands for the radial quantum number. From (11) it is clear that the leading contribution to $\mathscr{E}$ is of order $\bar{k}^2$. The next contribution, of order $\bar{k}$, is given in (14). It may be pointed out that although our expression (14) is identical to that obtained in [4], in our case ω contains non-relativistic as well as relativistic contributions as $U(r)$ in (9) contains terms of order $1/c^2$. This will be made more explicit when we present the results for the power-law potentials. Expanding all quantities in powers of $1/c^2$, it is possible to separate the contributions to ω:

$$\omega = \omega_{\mathrm{NR}} + \omega_{\mathrm{R}}. \tag{17}$$

Here ω_{NR} stands for the non-relativistic part independent of c and ω_{R} contains terms of order $1/c^2$ and higher.

We propose to determine the shift parameter a in such a manner that the non-relativistic part of (14) vanishes. This gives

$$a = 2 - 2(2n_r+1)\frac{m\omega_{\mathrm{NR}}}{\hbar} \tag{18}$$

just as in equation (15) of [4]. Consequently, the term of order $\bar{k}$ in (10) picks up only a contribution for the relativistic part of the bound-state energy. From (10), (11), (14), (17) and (18) we get finally

$$\begin{aligned} E = V(r_0) + mc^2\bigg(&1 + \frac{\hbar^2\bar{k}^2}{4m^2c^2r_0^2} + \frac{(1+2n_r)\hbar\omega_{\mathrm{R}}}{mc^2r_0^2}\bar{k} + \frac{2}{mc^2r_0^2}(b^{(0)}+b^{(1)}+b^{(2)}) \\ &+ \frac{2}{mc^2r_0^2\bar{k}}(c^{(1)}+c^{(2)}+c^{(3)}+c^{(4)}+\dots)\bigg)^{1/2}. \end{aligned} \tag{19}$$

For the potential in (1), we give here the expansions of the relevant quantities of (19) correct up to order $1/c^2$:

$$r_0 = x^{1/(\nu+2)}\left(1-\frac{\hbar^2\bar{k}^2}{8m^2c^2}\frac{1}{\nu+2}x^{-2/(\nu+2)}\right) \tag{20}$$

$$\omega = \frac{\hbar}{2m}\sqrt{\nu+2}\left(1-\frac{A}{2mc^2}\frac{\nu}{\nu+2}x^{\nu/(\nu+2)}\right) \tag{21}$$

$$a = 2-(1+2n_r)\sqrt{\nu+2} \tag{22}$$

$$\tilde{\varepsilon}_1 = \frac{\hbar^2}{2m}(1+2n_r)(\nu+2)^{1/4}\left(1+\frac{A}{4mc^2}\frac{\nu}{\nu+2}x^{\nu/(\nu+2)}\right) \tag{23a}$$

$$\tilde{\varepsilon}_2 = -\frac{3\hbar^2}{4m}(1+2n_r)\left(1+\frac{A}{2mc^2}\frac{\nu}{\nu+2}x^{\nu/(\nu+2)}\right) \tag{23b}$$

$$\tilde{\varepsilon}_3 = \frac{\hbar^2(\nu+2)(\nu-5)}{24m(\nu+2)^{3/4}}\left(1-\frac{3A}{4mc^2}\frac{\nu}{\nu+2}\frac{1+3\nu}{\nu-5}x^{\nu/(\nu+2)}\right) \tag{23c}$$

$$\tilde{\varepsilon}_4 = \frac{\hbar^2(\nu^2-8\nu+27)}{96m}\left(1+\frac{2A}{mc^2}\frac{\nu}{\nu+2}\frac{(\nu+1)(8-3\nu)}{\nu^2-8\nu+27}x^{\nu/(\nu+2)}\right) \tag{23d}$$

$$\tilde{\delta}_1 = -\frac{\hbar^2[(1+2n_r)^2(\nu+2)-1]}{4m(\nu+2)^{1/4}}\left(1+\frac{A}{4mc^2}\frac{\nu}{\nu+2}x^{\nu/(\nu+2)}\right) \tag{23e}$$

$$\tilde{\delta}_2 = \frac{3\hbar^2[(1+2n_r)^2(\nu+2)-1]}{8m(\nu+2)^{1/2}}\left(1+\frac{A}{2mc^2}\frac{\nu}{\nu+2}x^{\nu/(\nu+2)}\right) \tag{23f}$$

$$\tilde{\delta}_3 = \frac{\hbar^2(1+2n_r)}{m(\nu+2)^{1/4}}\left(1+\frac{3A}{4mc^2}\frac{\nu}{\nu+2}x^{\nu/(\nu+2)}\right) \tag{23g}$$

$$\tilde{\delta}_4 = -\frac{5\hbar^2(1+2n_r)}{4m(\nu+2)^{1/2}}\left(1+\frac{A}{mc^2}\frac{\nu}{\nu+2}x^{\nu/(\nu+2)}\right) \tag{23h}$$

$$\tilde{\delta}_5 = \frac{\hbar^2(\nu-7)(\nu^2-5\nu+24)}{480m(\nu+2)^{1/4}}\left(1-\frac{5A}{4mc^2}\frac{\nu}{\nu+2}\frac{11\nu^3-44\nu^2+25\nu+128}{(\nu-7)(\nu^2-5\nu+24)}x^{\nu/(\nu+2)}\right) \tag{23i}$$

$$\tilde{\delta}_6 = \frac{\hbar^2(\nu^4-17\nu^3+119\nu^2-463\nu+1200)}{2880m(\nu+2)^{1/2}}$$
$$\times\left(1-\frac{A\nu}{2mc^2(\nu+2)}\frac{59\nu^4-399\nu^3+833\nu^2+39\nu-3052}{\nu^4-17\nu^3+119\nu^2-463\nu+1200}x^{\nu/(\nu+2)}\right) \tag{23j}$$

where

$$x = \frac{\hbar^2\bar{k}^2}{4mA\nu}. \tag{24}$$

One may easily check that the first term of each of the expressions (20)-(23) corresponds to the non-relativistic result of Imbo *et al* [4].

3. Applications

3.1. Relativistic mass spectra of $q\bar{q}$ systems

In order to apply the relativistic $1/\bar{k}$ expansion illustrated in § 2 to the meson spectroscopy, one must be able to formulate the effective one-particle KG equation for a system of two particles. The relativistic wave equation for a system of two spin-zero particles of masses m_1 and m_2 bound to each other by a potential V is given by

$$(E-V)\psi=[(-c^2\hbar^2\nabla_1^2+m_1^2c^4)^{1/2}+(-c^2\hbar^2\nabla_2^2+m_2^2c^4)^{1/2}]\psi \tag{25}$$

where

$$\nabla_1^2=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial z_1^2} \qquad \nabla_2^2=\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial y_2^2}+\frac{\partial^2}{\partial z_2^2}.$$

Let us now shift the origin to the centre of mass $(\bar{x},\bar{y},\bar{z})$ of the system. Then the coordinates of the particles in the new frame in terms of the relative coordinates are

$$x_1'=\frac{m_2}{M}\xi \qquad y_1'=\frac{m_2}{M}\eta \qquad z_1'=\frac{m_2}{M}\zeta \tag{26a}$$

$$x_2'=-\frac{m_1}{M}\xi \qquad y_2'=-\frac{m_1}{M}\eta \qquad z_2'=-\frac{m_1}{M}\zeta \tag{26b}$$

where

$$\xi=x_1-x_2 \qquad \eta=y_1-y_2 \qquad \zeta=z_1-z_2 \qquad M=m_1+m_2.$$

Ignoring the motion of the centre of mass of the system one may write [12]

$$\nabla_i^2=\nabla_i'^2 \qquad i=1,2. \tag{27}$$

Use of (26) and (27) in (25) gives

$$(E-V)\psi=\left[\left(1+\frac{1}{\lambda}\right)[-c^2\hbar^2(1+\lambda)^2\nabla_\xi^2+m_1^2c^4]^{1/2}\right]\psi \tag{28}$$

where $\lambda=m_1/m_2$.

Since we are interested in studying the spectra and various other properties of quarkonia which are the bound states of identical quark and antiquark, we set $m_1=m_2=m$. Furthermore, changing over to the centre of mass coordinates, (28) become simpler. We thus get

$$(\tfrac{1}{2}E-\tfrac{1}{2}V)\psi=(-c^2\hbar^2\nabla^2+m^2c^4)^{1/2}\psi. \tag{29}$$

Comparing (29) and (2) we find that both the energy and the potential for the two-particle case have been scaled down by a factor of two. Taking care of these changes of numerical factors, we obtain the expression for the total mass of the quarkonium including relativistic effects up to order $1/c^2$:

$$\begin{aligned}M(n_r,l)=B+2mc^2&+\frac{\hbar^2}{m}\left(\frac{\hbar^2\bar{k}^2}{2mA\nu}\right)^{-2/(\nu+2)}\left(\frac{\nu+2}{4\nu}\bar{k}^2+\frac{Z_1}{72}+\frac{Z_2}{864\sqrt{\nu+2}}\frac{1}{\bar{k}}+\dots\right)\\&-\frac{1}{c^2}\frac{\hbar^4\bar{k}^4}{64m^3}\left(\frac{\hbar^2\bar{k}^2}{2mA\nu}\right)^{-4/(\nu+2)}\left(1+\frac{4(1+2n_r)}{\sqrt{\nu+2}}\frac{1}{\bar{k}}\right.\\&\left.+\frac{2Z_3+\nu Z_1}{9(\nu+2)}\frac{1}{\bar{k}^2}+\frac{3Z_4+\nu Z_2}{108(\nu+2)^{3/2}}\frac{1}{\bar{k}^3}+\dots\right)\end{aligned} \tag{30}$$

in which

$$
\begin{aligned}
&\bar{k}=1+2l+(1+2n_r)\sqrt{\nu+2}\\
&Z_1=(2-\nu)(\nu+1)(1+6n_r+6n_r^2)\\
&Z_2=(\nu+1)(\nu-2)[(\nu+1)(\nu-2)+(7\nu^2-31\nu-62)n_r+(5\nu^2-29\nu-58)(3n_r^2+2n_r^3)]\\
&Z_3=(2-11\nu-\nu^2)(1+6n_r+6n_r^2)+6\nu^2\\
&Z_4=(\nu^4-42\nu^3+101\nu^2-28\nu-28)+(7\nu^4-86\nu^3+419\nu^2-484\nu-132)n_r\\
&\qquad+(15\nu^4-6\nu^3+651\nu^2-1284\nu-228)n_r^2\\
&\qquad+(10\nu^4-4\nu^3+434\nu^2-856\nu-152)n_r^3.
\end{aligned}
\tag{31}
$$

The expression (30), comprising relativistic corrections to the non-relativistic limit, is useful in obtaining a quantitative understanding of the meson spectrum as well as allowing a comparison with other relativistic calculations. We obtain a satisfactory description of the meson spectroscopy with reasonable quark masses in the context of the three different potentials, linear, harmonic and Martin potentials, which are commonly used for the $q\bar{q}$ system.

For the sake of comparing our results with earlier calculations we take the following parameters and quark masses:

(i) Linear potential [12]:

$V(r)=B+Ar$ $\quad A=0.300\ \mathrm{GeV}^2$

$B_s=-1.080$ GeV $\quad B_c=-1.720$ GeV $\quad B_b=-1.490$ GeV

$m_s=0.475$ GeV $\quad m_c=2.000$ GeV $\quad m_b=5.174$ GeV.

(ii) Harmonic oscillator potential [13]:

$V(r)=B+Ar^2$ $\quad A=0.030\ \mathrm{GeV}^3$

$B_s=-0.636$ GeV $\quad B_c=-0.361$ GeV $\quad B_b=-1.000$ GeV

$m_s=0.518$ GeV $\quad m_c=1.566$ GeV $\quad m_b=5.174$ GeV.

(iii) Martin potential [14]:

$V(r)=B+Ar^{0.1}$

$A=6.8698$ and $B=-8.064$ (both in appropriate GeV units)

$m_s=0.518$ GeV $\quad m_c=1.8$ GeV $\quad m_b=5.174$ GeV.

Predictions of various mass spectra ensuing from the use of the linear, harmonic and Martin potentials, along with the respective values of previous calculations, are presented in table 1. For the linear potential, our results compare fairly well with those obtained by Kang and Schnitzer [12], who used an interpolation technique based on the WKB theory. For the harmonic oscillator potential, our results have been compared with those of Ram and Halasa [13]. In this case also, we observe better agreement of our results with experiment [15]. It may be mentioned here that in both cases the relativistic corrections are in general in the right directions. The corrections are typically within 5% for the c- and b-quark systems. For the strange quark mesons the relativistic correction increases sharply with the increase of n_r and l values and consequently our calculation up to order $1/c^2$ is not adequate. However, for the low-lying ϕ mesons we obtain reasonably nice results.

Table 1. Meson masses (in GeV) with and without relativistic corrections estimated from equation (30) for the linear, harmonic and Martin potentials. The predicted values of [12] and [13] are shown in parentheses and square brackets, respectively.

Quark system	States	n_r	l	Mass (GeV) Linear Non-relativistic	Linear Relativistic	Harmonic Non-relativistic	Harmonic Relativistic	Martin Non-relativistic	Martin Relativistic	Experiment [15]
	ϕ	0	0	1.222	1.030 (1.019)	1.126	1.020 [1.022]	0.964	0.815	1.020
$s\bar{s}$	ϕ (Average *P*-state)	0	1	1.814	1.488 (1.516)	1.610	1.363 [1.449]	1.425	1.299	1.440
	ϕ'	1	0	2.230	1.639 (1.806)	2.093	1.564 [1.800]	1.601	1.394	1.650
	ψ	0	0	3.117	3.100 (3.105)	3.188	3.176 [3.179]	3.067	3.029	3.097
	ψ (Average *P*-state)	0	1	3.484	3.454 (3.456)	3.467	3.440 [3.443]	3.503	3.470	3.494
$c\bar{c}$	ψ'	1	0	3.742	3.688 (3.696)	3.745	3.687 [3.695]	3.668	3.615	3.686
	ψ''	2	0	4.251	4.153 (4.169)	4.301	4.160 [4.186]	4.016	3.956	4.030
	ψ''	3	0	4.702	4.553 (4.580)	4.858	4.596 [4.656]	4.267	4.202	4.415
	Υ	0	0	9.468	9.464	9.577	9.576	9.446	9.434	9.461
	Υ (Average *P*-state)	0	1	9.735	9.729	9.730	9.728	9.860	9.850	9.888
$b\bar{b}$	Υ'	1	0	9.923	9.912	9.883	9.878	10.017	10.000	10.023
	Υ''	2	0	10.294	10.274	10.190	10.177	10.349	10.330	10.356
	Υ'''	3	0	10.622	10.591	10.496	10.472	10.586	10.566	10.578

For the Martin potential, the relativistic corrections slightly worsen the non-relativistic predictions. This is quite expected as the parameters of this potential were determined assuming that a non-relativistic treatment will fit the data best. It is then reasonable to do a relativistic calculation for the Martin-like potential provided one redetermines the parameters A and B by fitting the relativistic predictions with the experimental data.

3.2. Leptonic decay width

Since we are now dealing with the Schrödinger-like equation (8), we can also calculate the leptonic decay width of a heavy neutral vector meson using the familiar Van Royen-Weisskopf formula [16]

$$\Gamma(v \to l^+ l^-) = \frac{16\pi\hbar^3\alpha^2 e_q^2}{M_v^2} |\psi(0)|^2 \tag{32}$$

in which M_v is the mass of the meson and $\psi(0)$ is the total wavefunction of the composite system evaluated at the origin. $\psi(r, \hat{\theta}, \varphi)$ is related to the reduced radial wavefunction $u_{n_r,l}(r)$ in (12) by

$$\psi(r, \hat{\theta}, \varphi) = r^{(1-N)/2} u_{n_r l}(r) Y_l^m(\hat{\theta}, \varphi)$$

in which $Y_l^m(\hat{\theta}, \varphi)$ are the spherical harmonics (for $N = 3$).

We compute $|\psi(0)|^2$ for the s wave using the well known identity [16]

$$|\psi(0)|^2 = \frac{m}{2\pi\hbar^2}\left\langle\frac{\partial U}{\partial r}\right\rangle$$

and obtain finally

$$\begin{aligned}
|\psi(0)|^2 = {} & \frac{1}{8\pi} m^{3/(\nu+2)} \frac{S_1}{S_0} \tilde{\omega}^{(\nu-1)/\tilde{\omega}} \left(\frac{2A\nu}{\hbar^2}\right)^{3/(\nu+2)} \bar{k}^{(\nu-1)(2-\tilde{\omega})/(\nu+2)} \\
& \times \left[\!\left[1 + \frac{1}{c^2}\left(\frac{1}{m}\right)^{2(\nu+1/(\nu+2)} \left(\frac{\hbar^2\bar{k}^2}{2A\nu}\right)^{\nu/(\nu+2)} \right.\right. \\
& \left.\left. \times \left\{ \frac{A}{2}\left[1 - \frac{S_2}{S_1}\left(\frac{\bar{k}}{\tilde{\omega}}\right)^{-\nu/\tilde{\omega}}\right] + \frac{3}{4}\frac{A\nu}{\nu+2} \right\} \right]\!\right]
\end{aligned} \tag{33}$$

in which

$$\begin{aligned}
S_0 = {} & \sum_{j=0}^{n_r} (-1)^j \left(n_r + \frac{k-2}{\tilde{\omega}}\right)! \, \Gamma\left(j + \frac{k}{\tilde{\omega}}\right) \Gamma\left(1 + 2n_r - j - \frac{2}{\tilde{\omega}}\right) \\
& \times \left[n_r! j! (n_r - j)! \left(j + \frac{k-2}{\tilde{\omega}}\right)! \, \Gamma\left(1 - j - \frac{2}{\tilde{\omega}}\right)\right]^{-1}
\end{aligned} \tag{34a}$$

$$\begin{aligned}
S_1 = {} & \sum_{j=0}^{n_r} (-1)^j \left(n_r + \frac{k-2}{\tilde{\omega}}\right)! \, \Gamma\left(j + \frac{k+\nu-1}{\tilde{\omega}}\right) \Gamma\left(1 + n_r - j - \frac{\nu+1}{\tilde{\omega}}\right) \\
& \times \left[n_r! j! (n_r - j)! \left(j + \frac{k-2}{\tilde{\omega}}\right)! \, \Gamma\left(1 - j - \frac{\nu+1}{\tilde{\omega}}\right)\right]^{-1}
\end{aligned} \tag{34b}$$

$$\begin{aligned}
S_2 = {} & \sum_{j=0}^{n_r} (-1)^j \left(n_r + \frac{k-2}{\tilde{\omega}}\right)! \, \Gamma\left(j + \frac{k+2\nu-1}{\tilde{\omega}}\right) \Gamma\left(1 + n_r - j - \frac{2\nu+1}{\tilde{\omega}}\right) \\
& \times \left[n_r! j! (n_r - j)! \left(j + \frac{k-2}{\tilde{\omega}}\right)! \, \Gamma\left(1 - j - \frac{2\nu+1}{\tilde{\omega}}\right)\right]^{-1}.
\end{aligned} \tag{34c}$$

One must note that the relativistic corrections in our expressions come through the $1/c^2$ dependence of the effective potential $U(r)$ given in (9). Using (33) in (32), one may compute the leptonic decay width of heavy vector mesons, taking into account the relativistic effect.

Before proceeding further for numerical calculation, we would like to emphasise that for the linear and harmonic oscillator potentials ($\nu = 1$ and $\nu = 2$) the non-relativistic part of the decay width does not vary with the quantum number n_r, and consequently the relativistic part alone is not adequate to generate sufficient variation of the widths as required for different quantum states of the vector mesons. For this reason we give here the calculated ratios of the leptonic widths of ψ and γ systems for the Martin potential only and compare those with the available experimental values [15]. The results are

$$(2S/1S)_\psi = 0.423 \qquad \text{experimentally: } 0.45 \pm 0.06$$

$$(3S/1S)_\psi = 0.260 \qquad \text{experimentally: } 0.16 \pm 0.04$$

$$(2S/1S)_\gamma = 0.515 \qquad \text{experimentally: } 0.45 \pm 0.04$$

$$(3S/1S)_\gamma = 0.349 \qquad \text{experimentally: } 0.33 \pm 0.04.$$

This comparison provides a check of our relativistic leading-order bound-state wavefunctions. Except for the 3S/1S leptonic width ratios for charmonium, the agreement of our predicted results with the experimental values is reasonably good in view of the leading-order calculation. Further accuracy may be achieved by using non-leading terms in the wavefunction of [5]. However, that involves complicated analytic expressions which we avoid here.

4. Discussion

In this paper we have generalised the shifted large-N method (originally developed for the Schrödinger equation) to obtain the relativistic mass spectra and wavefunctions of mesons which are composite systems of quarks bound to each other by power-law type confining potentials. The main trick of our calculation lies in the manipulation of converting the KG equation to an effective Schrödinger-like equation. We have also given a prescription for the choice of the relativistic shift parameter. The formalism has been developed without sacrificing the elegance and accuracy of the non-relativistic results achieved previously by Sukhatme and his co-workers. Our relativistic shifted large-N expansion method not only gives consistent results for the mesonic systems, but has a major advantage over the previous relativistic calculations as it gives simple compact analytic expressions for the non-relativistic as well as for the relativistic parts of the energy eigenvalues and eigenfunctions. One may verify that numerical results can be obtained even with the help of a desk calculator. Furthermore, the method enables one to see clearly the algebraic factors responsible for individual contributions to the non-relativistic and relativistic components of a physical quantity.

As a final remark, we would like to mention that our method may be useful in obtaining analytic expression for relativistic single-electron (nl, $n'l'$) dipole transition-matrix elements for radiative transitions between atomic inner shells [17]. Results for various bound-bound transitions for atomic inner shells have recently been obtained using an analytic perturbation theory [18] for relativistic systems. Study of these parameters from the standpoint of the present non-perturbative relativistic approach may have significance in atomic phenomena. Work along these lines is in progress and will be reported later.

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